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Accelerated Life Testing Under Competing Exponential Failure Distributions

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Accelerated Life Testing Under Competing Exponential Failure Distributions

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Key Words and Phrases: accelerated life testing, competing risks, independent exponential lifetimes, survival function, mean time to failure, progressive censoring, type I censoring, type II censoring.

ABSTRACT

Accelerated life testing of a product is commonly used to estimate parameters of devices with extremely long lifetimes. The problem of analyzing accelerated life tests when the product can fail from any one of p independent causes is considered. For each component it is assumed that the time to failure distribution is exponential with a hazard rate which follows a "power rule." The method of maximum likelihood is used to obtain estimators of the power rule parameters when the life test is either type I, type II, or, progressively censored. Estimators of each component's mean failure time and survival function are obtained. Estimators of system or subsystem survival functions are also obtained. Results of a simulation study are presented.

1. Introduction

Accelerated life testing of a product is often used to obtain information on its performance under normal use conditions. Such testing involves subjecting test items to conditions more extreme than usually encountered in normal use. This results in decreasing the item's mean life and reduces test time and costs. The estimates obtained at these severe conditions are extrapolated to normal use conditions. This procedure is summarized in Chapter 9 of Mann, Schafer, and Singpurwalla (1974).

Little work has been done on analyzing accelerated life tests when an item can fail from any one of several distinct causes. Suppose an item consists of p components such that the failure of any one component causes the item to fail. For each item data consist of the time at which the item fails and knowledge of which component failed. The problem is to estimate the parameters of the failure distribution of each component at normal use conditions from data collected in an accelerated life test. When there are data at use conditions only, this problem has been considered by David and Moeschberger (1978), and Klabfleisch and Prentice (1980). Nelson (1974) has considered the accelerated life test problem with competing risks for normal populations.

Formally, let $V_1,\ldots,V_s$ be s stress levels at which the accelerated life test is conducted. At stress V_i, let X_{ij} denote the lifetime of the j^{th} component, $j = 1,\ldots,p$, $i = 1,\ldots,s$. Assume that the component life lengths are independent and each lifetime is exponentially distributed with survival function

$$P(X_{ij} > x) = \bar{F}_{ij}(x) = \exp(-\lambda_{ij}x),\ x > 0,\ \lambda_{ij} > 0, \qquad (1.1)$$
$$i = 1,\ldots,s$$
$$j = 1,\ldots,p$$

Assume that the stress and component life distributions are related by the power rule model,

$$\lambda_{ij} = \exp(A_j)V_i^{B_j}, \tag{1.2}$$

where A_j, B_j are constants to be estimated from the accelerated life test.

In Section 2 the method of maximum likelihood is used to obtain estimates of A_j and B_j for various censoring schemes. In Section 2.1 type I censoring is considered, Section 2.2 considers type II censoring, while Section 2.3 discusses progressive censoring.

The estimators of A_j and B_j obtained in Section 2 are used in Section 3 to obtain estimators of component mean survival time under normal use conditions. Confidence intervals and reduced bias estimators of these parameters are also obtained. Estimates of the component survival function and system survival function are obtained. The problem of estimating system life when a group of components are redesigned to increase reliability is also considered.

In Section 4 results of a simulation study are presented.

2. Maximum Likelihood Estimation of the Power Rule Parameters

Let $V_1, V_2, \ldots, V_s$ be the stress levels at which the accelerated life test is to be conducted. At each stress level we assume that the item survives until one of its p components fails. Let $X_{i1}, X_{i2}, \ldots, X_{ip}$ denote the life lengths of the p components of an item on test at stress V_i. Assume that the X_{ij}'s are independent exponential random variables with hazard rate (1.2).

$$\lambda_{ij} = V_i^{B_j} \exp(A_j),\ j = 1,\ldots,p,\ i = 1,\ldots,s. \tag{2.1}$$

For an item put on test at stress V_i we observe $Y_i = \text{minimum }(X_{i1},\ldots,X_{ip})$ and an indicator variable which describes which of the p components failed. The method of maximum likelihood is used to find estimators of A_j and B_j for various censoring schemes.

2.1 Type I Censoring

At each of the s stress levels, n_i items are put on test, $i = 1,\ldots,s$. Testing is continued until some fixed time τ_i. These τ_i's may be different for each stress level to allow for increased testing time at low stresses. Such a testing scheme is a type I censored accelerated life test.

At stress V_i suppose r_i items have failed prior to time τ_i, $i = 1,\ldots,s$. Let $Y_{i1}, Y_{i2},\ldots,Y_{ir_i}$ denote the corresponding failure times. That is, $Y_{ik} = \min(X_{i1k}, X_{i2k},\ldots,X_{ipk})$ where X_{ijk} is the lifetime of the j^{th} component of the k^{th} item which failed prior to time τ_i at stress V_i, $j = 1,\ldots,p$, $k = 1,\ldots,r_i$, $i = 1,\ldots,s$. Let m_{ij} denote the number of items which failed due to failure of component j, $j = 1,\ldots,p$. Note that $r_i = \sum_{j=1}^{p} m_{ij}$. Define the total time on test by

$$T_i = \sum_{k=1}^{r_i} Y_{ik} + (n_i - r_i)\tau_i,\ i = 1,\ldots,p. \tag{2.2}$$

The likelihood function of interest is proportional to the product of the likelihood at each stress level. That is,

$$L \propto \prod_{i=1}^{s} \left[\prod_{j=1}^{p} \lambda_{ij}^{m_{ij}} \exp(-\lambda_{ij} T_i) \right]. \tag{2.3}$$

Interchanging the order of multiplication we see that L factors into the product of p likelihoods, one for each component. That is,

$$L \propto \prod_{j=1}^{p} L_j \text{ where}$$

$$L_j \propto \prod_{i=1}^{s} \lambda_{ij}^{m_{ij}} \exp(-\lambda_{ij} T_i), \; i = 1, \ldots, p. \tag{2.4}$$

Hence, it suffices to maximize each component likelihood L_j individually. The resulting maximum likelihood estimators, $(\hat{A}_j, \hat{B}_j)$ will be asymptotically independent of $(\hat{A}_k, \hat{B}_k)$ when $j \neq k$ as all $n_i \to \infty$.

The likelihood equations are

$$\hat{A}_j = \ell n\left(\sum_{i=1}^{s} m_{ij}\right) - \ell n\left(\sum_{i=1}^{s} T_i V_i^{\hat{B}_j}\right), \; j = 1, \ldots, p. \tag{2.5}$$

and $$\left(\sum_{i=1}^{s} m_{ij} \ell n \, V_i\right)\left(\sum_{i=1}^{s} T_i V_i^{\hat{B}_j}\right) = \left(\sum_{i=1}^{s} m_{ij}\right)\left(\sum_{i=1}^{s} T_i V_i^{\hat{B}_j} \ell n \, V_i\right), \; j = 1, \ldots, p. \tag{2.6}$$

$\hat{B}_j$ can be obtained by solving (2.6) numerically using the Newton-Raphson procedure. Clearly $\hat{A}_j$ is determined by (2.5) once $\hat{B}_j$ is found.

From (2.4), the second derivitives are found to be

$$\frac{\delta^2 \ell n \, L_j}{\delta A_j^2} = -\sum_{i=1}^{s} T_i V_i^{B_j} \exp(A_j), \; j = 1, \ldots, p, \tag{2.7}$$

$$\frac{\delta^2 \ln L_j}{\delta A_j \delta B_j} = -\sum_{i=1}^{s} T_i V_i^{B_j} \exp(A_j) \ln V_i, \quad j = 1,\ldots,p, \qquad (2.8)$$

and

$$\frac{\delta^2 \ln L_j}{\delta B_j^2} = -\sum_{i=1}^{s} T_i V_i^{B_j} \exp(A_j) (\ln V_i)^2, \quad j = 1,\ldots,p. \qquad (2.9)$$

To find the information matrix, note that Y_{ik} has an exponential distribution with hazard rate $\lambda_i = \sum_{j=1}^{p} \lambda_{ij}$.

Writing, $E(Y_{ik}) = E(Y_{ik}|Z_{ik} = 1)P(Z_{ik} = 1) + E(Y_{ik}|Z_{ik} = 0)P(Z_{ik} = 0)$, where $Z_{ik} = 1$ if $Y_{ik} \leq \tau_i$ and $Z_{ik} = 0$ if $Y_{ik} > \tau_i$, it can be shown that $E(T_i) = n_i \Pi_i / \lambda_i$ where $\Pi_i = 1 - \exp(-\lambda_i \tau_i)$, $i = 1,\ldots,s$ is the probability an item will fail before time τ_i. The inverse of the information matrix is

$$\Sigma_j = \frac{1}{D_j} \begin{bmatrix} \sum_{i=1}^{s} \frac{\lambda_{ij} n_i \Pi_i}{\lambda_i} (\ln V_i)^2 & -\sum_{i=1}^{s} \frac{\lambda_{ij} n_i \Pi_i}{\lambda_i} \ln V_i \\ -\sum_{i=1}^{s} \frac{\lambda_{ij} n_i \Pi_i}{\lambda_i} \ln V_i & \sum_{i=1}^{s} \frac{\lambda_{ij} n_i \Pi_i}{\lambda_i} \end{bmatrix}$$

where

$$D_j = \sum_{i \neq k}^{s} \sum^{s} \frac{\lambda_{ij} \lambda_{kj} n_i n_k \Pi_i \Pi_k}{\lambda_i \lambda_k} \ln V_i \ln(V_i / V_k), \quad j = 1,\ldots,p. \qquad (2.10)$$

From David and Moeschberger (1978) note that the maximum likelihood estimator of λ_{ij} is $\frac{m_{ij}}{T_i}$ and of λ_i is $\frac{r_i}{T_i}$. A consistant estimator of Σ_j is

$$\hat{\Sigma}_j = \frac{1}{\hat{D}_j}\begin{bmatrix} \sum_{i=1}^{s} m_{ij}(\ell n\, V_i)^2 & -\sum_{i=1}^{s} m_{ij}\,\ell n\, V_i \\ -\sum_{i=1}^{s} m_{ij}\,\ell n\, V_i & \sum_{i=1}^{s} m_{ij} \end{bmatrix}$$

where

$$\hat{D}_j = \sum_{i\neq k}^{s}\sum^{s} m_{ij} m_{kj}\,\ell n\, V_i\,\ell n(V_i/V_k),\ j = 1,\ldots,p. \tag{2.11}$$

2.2 Type II censoring

For type II censoring testing continues until a preassigned number of failures is observed. At each stress level, V_i, $i = 1,\ldots,s$, n_i items are put on test. Let Y_{ik} denote the failure time of the k^{th} item put on test, that is, $Y_{ik} = \min_{j=1,\ldots,p}(X_{ijk})$ where X_{ijk} is the life length of the j^{th} component of the k^{th} item put on test at stress V_i, $j = 1,\ldots,p$, $k = 1,\ldots,n_i$, $i = 1,\ldots,s$. For this testing scheme, testing at stress V_i continues until a preassigned number, r_i, of items has failed. Let $Y_{i(1)},\ldots,Y_{i(r_i)}$ be the ordered failure times of the r_i failures. Let m_{ij} denote the number of items which fail due to failure of component j, $j = 1,\ldots,p$. For type II censoring the total time on test is

$$T_i = \sum_{k=1} Y_{i(k)} + (n_i - r_i)\, Y_{i(r_i)},\ i = 1,\ldots,s. \tag{2.12}$$

The likelihood function of interest for type II censoring is identical to (2.3) with T_i as in (2.11). Hence, A_j and B_j can be obtained as in Section 2.1.

To find the information matrix, note that the second derivatives of L_j are as in 2.7, 2.8, and 2.9. Hence

$$\Sigma_j = \frac{1}{D_j}\begin{vmatrix} \sum_{i=1}^{s} \frac{\lambda_{ij}}{\lambda_i} r_i (\ln V_i)^2 & -\sum_{i=1}^{s} \frac{\lambda_{ij}}{\lambda_i} r_i \ln V_i \\ -\sum_{i=1}^{s} \frac{\lambda_{ij}}{\lambda_i} r_i \ln V_i & \sum_{i=1}^{s} \frac{\lambda_{ij}}{\lambda_i} r_i \end{vmatrix}$$

where

$$D_j = \sum_{i \neq k}\sum \frac{\lambda_{ij}\lambda_{kj} r_i r_k}{\lambda_i \lambda_k} \ln V_i \ln(V_i/V_k), \; j = 1,\ldots,p. \qquad (2.13)$$

The maximum likelihood estimators of λ_i and λ_{ij} are $\hat{\lambda}_{ij} = \frac{m_{ij}}{T_i}$ and $\hat{\lambda}_i = \frac{r_i}{T_i}$, respectively. Substituting these values in (2.12) the maximum likelihood estimator of Σ_j is obtained. This estimator is given by (2.11).

2.3 Progressive Censoring

Progressive censoring is a useful scheme to reduce test time and cost while still including some observations with extremely long life times. In this scheme, at several fixed times some fixed number of items are removed from study. This scheme is discussed in Cohen (1963) for the case of normal and exponential data from a population with a single failure mode when there is not an accelerated life test.

At stress level V_i, $i = 1,\ldots,s$ suppose that N_i items are put on test. Censoring will occur in M_i stages. At times $\tau_{i1}, \tau_{i2},\ldots,\tau_{iM_i-1}$ a fixed number r_{ik} items are removed from the accelerated life test.

At time τ_{iM_i} the test is either terminated with a random number r_{iM_i} items still functioning or a fixed number r_{iM_i} items are removed. We assume that N_i is sufficiently large so that there is at least r_{ik}, $k = 1,\ldots,M_i$ items surviving to time τ_{ik} to be censored. Let m_{ij} be the number of items which fail due to failure of component j, $j = 1,\ldots,p$, and, let $n_i = \sum_{j=1}^{p} m_{ij}$ be the total number of failures. Let $Y_{i1},\ldots,Y_{in_i}$ be the failure times of the n_i failures. As before, $Y_{ik} = \min(X_{i1k},\ldots,X_{ipk})$ where X_{ijk} is the lifetime of the j^{th} component of the k^{th} item which fails at stress V_i, $j = 1,\ldots,p$; $k = 1,\ldots,n_i$, $i = 1,\ldots,s$. For this scheme the total time on test is defined by

$$T_i = \sum_{k=1}^{n_i} Y_{ik} + \sum_{k=1}^{M_i} r_{ik}\tau_{ik}, \quad i = 1,\ldots,s. \tag{2.14}$$

As before it can be shown that the likelihood of interest is as in (2.4). The maximum likelihood estimators of A_j and B_j as well as the second derivatives are as in Section 2.1.

For this censoring scheme the information matrix and an estimator of the matrix are obtained in the Appendix.

3. Estimation of population parameters at use conditions

Suppose an accelerated life test has been conducted by one of the censoring schemes discussed in Section 2. Let $\hat{A}_j$ and $\hat{B}_j$ be the maximum likelihood estimators of A_j and B_j, $j = 1,\ldots,p$. Let

$$\Sigma_j = \begin{pmatrix} \sigma_{11}^{(j)} & \sigma_{12}^{(j)} \\ \sigma_{12}^{(ij)} & \sigma_{22}^{(j)} \end{pmatrix} \tag{3.1}$$

be the inverse of information matrix obtained from (2.10), (2.13) or A.10. Let $\hat{\Sigma}_j$ be the corresponding estimator of Σ_j obtained from

(2.11), A.11, or A.12. Based on this information we wish to estimate each component's mean survival time and survival function at the use conditions. In addition we shall obtain estimators of the survival function for the entire system of components or a subsystem of components under use conditions.

Let V_u denote the stress under use conditions. Let λ_{uj} and μ_{uj} denote the hazard rate and mean survival time at this stress. By (2.1)

$$\lambda_{uj} = V_u^{B_j} \exp A_j, \quad j = 1, \ldots, p \tag{3.2}$$

and

$$\mu_{uj} = V_u^{-B_j} \exp(-A_j), \quad j = 1, \ldots, p. \tag{3.3}$$

By the invariance principle of maximum likelihood estimators the maximum likelihood estimators of λ_{uj} and μ_{uj} are

$$\hat{\lambda}_{uj} = V_u^{\hat{B}_j} \exp(\hat{A}_j), \quad j = 1, \ldots, p. \tag{3.4}$$

and

$$\hat{u}_{uj} = V_u^{-\hat{B}_j} \exp(-\hat{A}_j), \quad j = 1, \ldots, p, \tag{3.5}$$

respectively. For large sample sizes at all the stress levels, $(\hat{A}_j, \hat{B}_j)$ are asymptotically bivariate normally distributed with mean vector (A_j, B_j) and variance Σ_j, $j = 1, \ldots, p$. Since $\ell n \hat{\lambda}_{uj}$ is linear in $\hat{A}_j$ and $\hat{B}_j$, it has an asymptotic normal distribution with mean $\ell n\, \lambda_{uj}$ and variance

$$\sigma_{uj}^2 = \sigma_{11}^{(j)} + \sigma_{22}^{(j)} (\ell n\, V_u)^2 + 2\sigma_{12}^{(j)}\, \ell n\, V_u. \tag{3.6}$$

Similarly, $\ln \hat{\mu}_{uj}$ is asymptotically normal with mean $\ln \mu_{uj}$ and variance σ_{uj}^2.

By using the properties of the log normal distribution one can show that asymptotically

$$E(\hat{\mu}_{uj}) = \mu_{uj} \exp(\sigma_{uj}^2/2),\ j = 1,\ldots,p \tag{3.7}$$

and the mean squared error of $\hat{\mu}_{uj}$ is $(E(\hat{\mu}_{uj} - \mu_{uj})^2)$ is

$$\text{m.s.e.}(\hat{\mu}_{uj}) = \mu_{uj}^2[\exp(2\sigma_{uj}^2) - 2\exp(\sigma_{uj}^2/2)\ 1],\ j = 1,\ldots,p, \tag{3.8}$$

with similar expressions for $\hat{\lambda}_{ui}$. From (3.7) we see that an unbiased estimator of μ_{uj} is

$$\tilde{\mu}_{uj} = \hat{\mu}_{uj} \exp(-\sigma_{uj}^2/2),\ j = 1,\ldots,p \tag{3.9}$$

which has mean squared error

$$\text{m.s.e.}(\tilde{\mu}_{uj}) = \mu_{uj}^2(\exp(\sigma_{uj}^2) - 1),\ j = 1,\ldots,p \tag{3.10}$$

which is always smaller than the mean squared error of $\hat{\mu}_{uj}$. Since σ_{uj}^2 is unknown a reduced bias estimater of μ_{uj} is obtained by substituting $\hat{\sigma}_{uj}^2$ in (3.9). Similar results hold for estimating λ_{uj}.

The asymptotic normality of $\ln \hat{\mu}_{uj}$ can be used to construct $(1 - \alpha) \times 100\%$ confidence intervals for μ_{uj}. These are

$$(\hat{\mu}_{uj} \exp(-Z_{1-\alpha/2}\sigma_{uj}),\ \hat{\mu}_{uj} \exp(Z_{1-\alpha/2}\sigma_{uj}),\ j = 1,\ldots,p \tag{3.11}$$

where $Z_{1-\alpha/2}$ is the $1 - \alpha/2^{th}$ percentage point of a standard normal random variable. A similar interval can be obtained for λ_{uj}.

Let $\bar{F}_{uj}(t) = \exp(-t\lambda_{uj})$ be the survival function of component j, $j = 1,\ldots,p$. The maximum likelihood estimator of $\bar{F}_{uj}(t)$ is

$$\hat{\bar{F}}_{uj}(t) = \exp(-t\hat{\lambda}_{uj}),\ t \geq 0,\ j = 1,\ldots,p. \tag{3.12}$$

An approximate 100 × (1 - α)% confidence interval for $\bar{F}_{uj}(t)$ is given by

$$\left(\hat{\bar{F}}_{uj}(t)^{\exp(Z_{1-\alpha/2}\,\sigma_{uj})},\ \hat{\bar{F}}_{uj}(t)^{\exp(-\,Z_{1-\alpha/2}\,\sigma_{uj})}\right) j = 1,\ldots,p. \quad (3.13)$$

Let K be a subset of $1,\ldots,p$ of cardinality K. We are interested in obtaining estimates of

$$\bar{F}_u^{(K)}(t) = \prod_{j\in K} \bar{F}_{uj}(t), \quad (3.14)$$

the survival function of an item which can fail only by the failure of components indexed by K. When $K = \{1,\ldots,p\}$ this is the survival function of an item of interest. When K is a proper subset of $\{1,\ldots,p\}$ then (3.14) represents the survival function of an item which has been redesigned so that those components indexed by K^c can not fail or have extremely long lifetimes.

From (3.12) the maximum likelihood estimator of $\bar{F}_u^{(K)}(t)$ is

$$\hat{\bar{F}}_u^{(K)}(t) = \prod_{j\in K} \hat{\bar{F}}_{uj}(t). \quad (3.15)$$

An approximate (1 - α) × 100% confidence interval for $\bar{F}_u^{(K)}$ is given by

$$\left(\prod_{j\in K} \hat{\bar{F}}_{ju}^{\exp(\sigma_{uj}Z_\beta)},\ \prod_{j\in K} \hat{\bar{F}}_{ju}^{\exp(-\,\sigma_{uj}Z_\beta)}\right) \quad (3.16)$$

where

$$\beta = \frac{1 + (1-\alpha)^{1/K}}{2}.$$

This is a conservative interval in the following sense. From (3.13),

$$(1-\alpha)^{1/P} = P\left(\hat{\bar{F}}_{uj}(t)^{\exp(Z_\beta \sigma_{uj})} \le \bar{F}_{uj}(t) \le \bar{F}_{ij(t)}^{\exp(-z_\beta \sigma_{uj})}\right), \text{ for } j = 1,\ldots,p.$$

Since $(\bar{F}_{ij}(t), \bar{F}_{uj}(t))$ are asymptotically independent for $j \neq j'$,

$$1-\alpha = P\left(\hat{\bar{F}}_{uj}(t)^{\exp(Z_\beta \sigma_{uj})} \le \bar{F}_{uj}(t) \le \bar{F}_{uj}(t)^{\exp(-Z_\beta \sigma_{uj})} ; \text{ for all } j\varepsilon K\right)$$
$$\le P\left(\prod_{j\varepsilon K} \bar{F}_{uj}(t)^{\exp(Z_\beta \sigma_{uj})} \le \bar{F}_u^{(K)}(t) \le \prod_{j\varepsilon K} \hat{\bar{F}}_{uj}(t)^{\exp(-Z_\beta \sigma_{uj})}\right).$$

4. Simulation Study

A simulation study was conducted to compare the effects of censoring on the estimation procedure. A two component series system of independent exponential components was used. An accelerated life test was conducted at 3 stress levels, $V_1 = 10$, $V_2 = 15$, $V_3 = 20$. At each stress level the observations were generated from an exponential distribution with hazard rate λ_{ij}, $i = 1,\ldots,3$ $j = 1,2$, where λ_{ij} is given by (1.2) with $A_1 = -1$, $B_1 = 1$, $A_2 = -3$, and $B_2 = 2$ throughout the study. The values of λ_{ij}, and $\eta_j = P$ (component j fails first) are given in Table I for V_1, V_2, V_3 and a usage stress of 5.

Table 1

V_i	λ_{i1}	λ_{i2}	η_1	η_2
5	1.8394	1.2447	.5964	.4036
10	3.6788	4.9787	.4249	.5751
15	5.5182	11.2021	.3300	.6700
20	7.3576	19.9148	.2698	.7302

For each censoring scheme 500 repetitions of the accelerated life test were performed. Maximum likelihood estimates of A_j, and

B_j, $j = 1, 2$ were found by using the Newton-Raphson procedure discussed previously. The estimated covariance matrix was found by appropriate expression. Both maximum likelihood and reduced bias estimators of the component hazard rate and mean survival at a use stress of 5 were calculated. Tables 2-7 report for each estimator its estimated mean (standard error), estimated bias (|Bias|/Standard error), estimated variance, and the P-value of a Kolmogorov-Smirnov test for normality for each component. The following censoring schemes were used:

Scheme I - complete failure times on all items, sample sizes $n_i = 20$, $i = 1, 2, 3$.

Scheme II - complete failure times on all items, sample sizes $n_i = 10$, $i = 1, 2, 3$.

Scheme III - type II censoring $n_i = 40$, $r_i = 20$, $i = 1, 2, 3$.

Scheme IV - type II censoring, $n_i = 40$, $r_i = 10$, $i = 1, 2, 3$.

Scheme V - type I censoring at the same fixed point $n_i = 40$, $i = 1, 2, 3$, $\tau_1 = .08$, $\tau_2 = .04$, $\tau_3 = .025$, $E(r_i) = 19.989$, $E(r_2) = 19.507$, $E(r_3) = 19.772$.

Scheme VI - type I censoring at the same fixed point $n_i = 40$, $i = 1, 2, 3$, $\tau_1 = .03$, $\tau_2 = .017$, $\tau_3 = .010$, $E(r_1) = 9.15$, $E(r_2) = 9.89$, $E(r_3) = 9.55$.

Several conclusions may be drawn from the study. First the asymptotic normality property of the m.l.e. estimators appears to hold for these moderate sample sizes. Secondly, the reduced [illegible] estimators $\tilde{\lambda}_{uj}$ and $\tilde{\mu}_{uj}$ perform as expected when the estimated variance was used. Finally, the asymptotic unbiasedness of the maximum likelihood estimators seems to be reasonable for these moderate sample sizes.

5. Dependent Risks

The assumption of independence of the times to failure of each component made in the previous sections can be replaced by the assumption that the failure times of the item follow a Marshall and Olkin (1967) multivariate exponential distribution. For this model simultaneous failures of several components is possible. Here there are $2^p - 1$ possible failure modes which represent system failure from the simultaneous failures of any subset of components. Each of these failure modes is assumed to have a time to occurrence distribution given by 1.1 with hazard rate 1.2. At use conditions the survival function of the time to failure distribution of a particular component can be estimated as in the previous section.

Bibliography

Cohen, A. C. (1963). Progressively censored samples in life testing," Technometrics 5, 327-339.

David, H. A. and Moeschberger, M. L. (1978). The Theory of Competing Risks. New York: MacMillan Publishing Co., Inc.

Kalbfleisch, J. D. and Prentice, R. L. (1980). The Statistical Analysis of Failure Time Data, John Wiley and Sons.

Mann, N. R., Schafer, R. E., and Singpurwalla, N. D. (1974). Methods for Statistical Analysis of Reliability and Life Data.

Marshall, A. W. and Olkin, I. (1967). A multivariate exponential distribution," J. Amer. Statist. Assoc. 62, 30-44.

Nelson, W. (1974). "Analysis of Accelerated Life Test Data With a Mix of Failure Modes by Maximum Likelihood," General Electric CR & D TIS Report 74 CRD 160, Schenectady, N. Y.

Appendix. Calculation of the information matrix for progressively censored data

The distribution of the total time on test T_i given by (2.13) depends only upon the distribution of the minimum of p lifetimes of the components which is exponential with parameter λ_i. To simplify the notation we shall suppress the subscript i which refers to the stress level under consideration. The problem is to calculate the expected value of the total time on test T given by

$$\sum_{i=1}^{n} Y_i + \sum_{i=1}^{M} r_i \tau_i . \tag{A.1}$$

Let f_i be the number of items which fail in the interval $[\tau_{i-1}, \tau_i)$, $i = 1,\ldots,M$ where $\tau_0 = 0$. Let S_i be the sum of the f_i failure times of those items failing in $[\tau_{i-1}, \tau_i)$, $i = 1,\ldots,M$. Let $F_i = 1 - \exp(-\lambda\tau_i)$ and $\bar{F}_i = 1 - F_i$, $i = 1,\ldots,M$. We consider two cases.

Case 1. r_M fixed

Suppose r_M is fixed so that $f_{M+1} = N - \sum_{i=1}^{M} (r_i + f_i)$ items fail in the interval (τ_m, ∞). Cohen (1963) shows that

$$E(f_i) = \begin{cases} N F_1 \\ \\ \left(N - \sum_{j=1}^{i-1} \frac{r_j}{\bar{F}_j}\right)(F_i - F_{i-1}), \quad i = 2,\ldots,M+1 \end{cases} \tag{A.2}$$

where $F_{M+1} = 1$.

To find $E(S_i)$ let $Y_{i1},\ldots,Y_{if_i}$ be the failure times of the f_i items which fail in the interval $[\tau_{i-1}, \tau_i)$, so that

$$S_i = \sum_{K=1}^{f_i} Y_{iK} \text{ for } i = 1,\ldots,M.$$

$$E(S_i) = EE\left(\sum_{K=1}^{f_i} Y_{iK} \middle| f_i\right) \tag{A.3}$$

$$= E(f_i E(Y_{i1} | \tau_{i-1} \leq Y_{i1} < \tau_i).$$

Now, given that Y_{i1} fails in $[\tau_{i-1}, \tau_i)$ the conditional expectation is

$$E(Y_{i1} | \tau_{i-1} \leq Y_{i1} < \tau_i) = \int_{\tau_{i-1}}^{\tau_i} y\lambda e^{-\lambda y} dy/(F_i - F_{i-1})$$

$$= (\lambda\tau_{i-1}\bar{F}_{i-1} - \lambda\tau_i\bar{F}_i + F_i - F_{i-1})/\lambda(F_i - F_{i-1}). \tag{A.4}$$

Combining A.2 and A.4 by A.3 we see that

$$E(S_i) = \begin{cases} N\left(\frac{F_1}{\lambda} - \tau_1\bar{F}_1\right) & \text{for } i = 1 \\ \left(N - \sum_{j=1}^{i-1} \frac{r_j}{\bar{F}_j}\right)\left(\tau_{i-1}\bar{F}_{i-1} - \tau_i\bar{F}_i + \frac{F_i - F_{i-1}}{\lambda}\right) & \text{for } i = 2,\ldots,M+1. \end{cases} \tag{A.5}$$

So

$$E\left(\sum_{i=1}^{M+1} S_i\right) = \frac{1}{\lambda}\left(N - \sum_{i=1}^{M} r_i - \sum_{i=1}^{m} r_i T_i\right) \tag{A.6}$$

and

$$E(T) = \frac{N - \sum_{i=1}^{M} r_i}{\lambda}. \qquad \text{(A.7)}$$

Case 2. r_M random

As in Case 1 the $E(f_i)$ is given by A.2 for $i = 1,\ldots,M$ and $E(S_i)$ is given by A.5 for $i = 1,\ldots,M$. It remains to find $E(r_M)$. Note that

$$r_M = N - \sum_{i=1}^{M} f_i - \sum_{i=1}^{M-1} r_i .$$

Hence

$$E(r_M) = N - \sum_{i=1}^{M} E(f_i) - \sum_{i=1}^{M-1} r_i ,$$

which is, from A.2,

$$E(r_M) = (N - \sum_{i=1}^{M-1} \frac{r_i}{\bar{F}_i})\bar{F}_K . \qquad \text{(A.8)}$$

Combining A.2, A.5, and A.8 we obtain

$$E(T) = \frac{N}{\lambda}\left(1 - e^{-\lambda\tau_M}\right) - \frac{\sum_{i=1}^{M-1} r_i}{\lambda}\left(1 - e^{-\lambda(\tau_M - \tau_i)}\right). \qquad \text{(A.9)}$$

The information matrix of interest for the accelerated life test, using the notation of Section 2, is

$$\Sigma_j = \frac{1}{D_j}\begin{vmatrix} \sum_{i=1}^{s} \lambda_{ij} (\ln V_i)^2 E(T_i) & -\sum_{i=1}^{s} \lambda_{ij} \ln V_i \, E(T_i) \\ -\sum_{i=1}^{s} \lambda_{ij} \ln V_i \, E(T_i) & \sum_{i=1}^{s} \lambda_{ij} E(T_i) \end{vmatrix} \tag{A.10}$$

where

$$D_j = \sum_{i \neq K}\sum \lambda_{ij}\lambda_{iK} E(T_i) E(T_K) \ln V_i \ln (V_i/V_K), \; j = 1, \ldots, P$$

and $E(T_i)$ is given by A.7 or A.9.

To estimate Σ_j one can show that for both r_{Mi} fixed or random the maximum likelihood estimators of λ_{ij} and λ_i are $\hat{\lambda}_j = \frac{m_{ij}}{T_i}$ and $\hat{\lambda}_i = \frac{n_i}{T_i}$, respectively. Hence when r_{Mi} is fixed the maximum likelihood estimator of Σ_j is

$$\hat{\Sigma}_j = \frac{1}{\hat{D}_j}\begin{vmatrix} \sum_{i=1}^{s} m_{ij} (\ln V_i)^2 & -\sum_{i=1}^{s} m_{ij} \ln V_i \\ -\sum_{i=1}^{s} m_{ij} \; n V_i & \sum_{i=1}^{s} m_{ij} \end{vmatrix} \tag{A.11}$$

where

$$\hat{D}_j = \sum_{i \neq K}\sum m_{ij} m_{iK} \ln V_i \ln(V_i/V_K), j = 1, \ldots, P.$$

When r_{M_i} is random the maximum likelihood estimator of Σ_j is

$$\hat{\Sigma}_j = \frac{1}{\hat{D}_j} \begin{vmatrix} \sum_{i=1}^{s} \frac{m_{ij}}{n_i} \hat{E}(n_i)(\ell n\, V_i)^2 & -\sum_{i=1}^{s} \frac{m_{ij}}{n_i} \hat{E}(n_i)\ell n\, V_i \\ -\sum_{i=1}^{s} \frac{m_{ij}}{n_i} \hat{E}(n_i)\ell n\, V_i & \sum_{i=1}^{s} \frac{m_{ij}}{n_i} \hat{E}(n_i) \end{vmatrix} \tag{A.12}$$

where

$$\hat{D}_j = \sum_{i \neq K} \sum \frac{m_{ij} m_{Kj}}{n_i n_K} \hat{E}(n_i)\hat{E}(n_K)\ell n\, V_i\, \ell n(V_i/V_K),\ j = 1,\ldots,p$$

and

$$\hat{E}(n_i) = N_i(1 - \exp(-\hat{\lambda}_i \tau_{M_i})) - \sum_{\ell=1}^{M_{i-1}} r_{i\ell}(1 - \exp(-\hat{\lambda}_i(\tau_{iM_i} - \tau_{i\ell}))),$$

$$i = 1,\ldots,s.$$

Table 2

Censoring Scheme I

	Component I					Component 2				
	True Value of Parameter	Mean	Bias	Var	Normal	True Value of Parameter	Mean	Bias	Var	Normal
$\hat{A}$	-1	-1.0207 (.0951)	.0207 (.22)	4.523	.996	-3	-3.0762 (.0726)	.0762 (.98)	2.6367	1.0
$\hat{B}$	1	1.0047 (.365)	.0047 (.13)	.666	.870	2	2.0277 (.0268)	.0227 (1.04)	.3586	1.0
$\hat{V}(\hat{A})$	4.2543	4.5557 (.0556)	.3013 (5.42)	1.5499	.0001	2.3750	2.4445 (.0170)	.0695 (4.95)	.1439	.0001
$\hat{V}(\hat{B})$	.6143	.6599 (.0085)	.0456 (5.31)	.0359	.0001	.3232	.3322 (.0022)	.0090 (4.20)	.0023	.0001
$\hat{V}(\hat{B}, \hat{A})$	-1.6073	-1.724 (.0215)	.1166 (5.42)	.2324	.0001	-.8714	-.8964 (.0022)	.0249 (4.15)	.0181	.0001
$\ln\hat{\lambda}_u$	.6094	.5962 (.0373)	.0132 (.35)	.6959	.672	.2189	.1873 (.03)	.0315 (1.05)	.4500	1.0
$\hat{\lambda}_u$	1.8394	2.5703 (.1157)	.7309 (6.32)	6.6902	.0001	1.2447	1.511 (.0514)	.2663 (5.18)	1.3211	.0001
$\tilde{\lambda}_u$	1.8394	1.8275 (.0827)	.0119 (.14)	3.414	.0001	1.2447	1.2389 (.0432)	.0058 (.1336)	.9315	.0001
$\hat{\mu}_u$	.5437	.7902 (.0410)	.2466 (6.02)	.8392	.0001	.8034	1.0363 (.0339)	.2329 (6.88)	.5737	.0001
$\tilde{\mu}_u$	.5437	.5366 (.0257)	.0071 (.28)	.3293	.0001	.8034	.8293 (.026)	.0259 (.995)	.3385	.0001
$\hat{\sigma}^2_u$	.6717	.7158 (.0085)	.0440 (5.18)	.0362	.0001	.3891	.4197 (.0032)	.0306 (9.59)	.0051	.0001

Table 3

Censoring Scheme II

	Component 1					Component 2				
	True Value of Parameter	Mean	Bias	Var	Normal	True Value of Parameter	Mean	Bias	Var	Normal
$\hat{A}$	-1.0	-1.132 (.1393)	.132 (.95)	9.6796	.968	-3	-3.0195 (.0983)	.0195 (.2)	4.8315	.[illegible]38
$\hat{B}$	1.0	1.0376 (.0531)	.0376 (.71)	1.4061	.880	2.0	2.0179 (.0363)	.0179 (.49)	.6586	.971
$\hat{V}(\hat{A})$	8.5087	10.169 (.2293)	1.6603 (7.24)	26.2856	.0001	4.750	4.9798 (.0483)	.2298 (4.76)	1.1670	.0001
$\hat{V}(\hat{B})$	1.2287	1.4701 (.0322)	.2414 (7.27)	.5195	.0001	.6464	.6774 (.0061)	.0310 (5.08)	.0188	.0001
$\hat{V}(\hat{B}, \hat{A})$	-3.2150	-3.8444 (.0852)	-.6294 (7.39)	3.6331	.0001	-1.7430	-1.8270 (.0171)	.0840 (4.9)	.1470	.0001
$\ln \hat{\lambda}_u$	.6094	.5379 (.0051)	.0715 (1.30)	1.5162	1.0	.2189	.2281 (.0409)	.0092 (.22)	.8198	.796
$\hat{\lambda}_u$	1.8394	3.6161 (.2958)	1.7767 (6.01)	43.6677	.0001	1.2447	1.8955 (.1020)	.6508 (6.38)	5.2024	.0001
$\tilde{\lambda}_u$	1.8394	1.7695 (.1403)	.0699 (.5)	9.8243	.0001	1.2447	1.2822 (.0700)	.0375 (.54)	2.4488	.0001
$\hat{\mu}_u$	.5437	3.1513 (2.0043)	2.6076 (1.3)	2004.65	.0001	.8034	1.2741 (.1081)	.4707 (4.35)	5.8434	.0001
$\tilde{\mu}_u$	.5437	.4647 (.0259)	.0790 (3.05)	.3368	.0001	.8034	.7718 (.0489)	.0316 (.65)	1.1955	.0001
$\hat{\sigma}^2_u$	1.3435	1.6023 (.03937)	.2588 (6.52)	.07748	.0001	.7780	.8536 (.0091)	.0756 (8.31)	.0413	.0001

Table 4

Censoring Scheme III

	Component 1					Component 2				
	True Value of Para-meter	Mean	Bias	Var	Nor-mal	True Value of Para-meter	Mean	Bias	Var	Nor-mal
$\hat{A}$	-1.0	- .9954 (.0989)	.0046 (.05)	4.8872	.97	-3	-3.1052 (.0661)	.1052 (1.59)	2.1859	1.0
$\hat{B}$	1.0	.9962 (.0377)	.0338 (.90)	.7107	.38	2	2.0236 (.0245)	.0236 (.97)	.2991	1.0
$\hat{V}(\hat{A})$	4.2544	4.6473 (.0603)	.3929 (6.55)	1.8201	.0001	2.375	2.421 (.0160)	.0457 (2.86)	.1279	.0002
$\hat{V}(\hat{B})$	.6143	.6742 (.0090)	.0599 (6.68)	.0401	.0011	.3232	.3289 (.002)	.0057 (2.85)	.0020	.001
$\hat{V}(\hat{B}, \hat{A})$	-1.607	-1.7599 (.0231)	.1526 (6.63)	.2661	.0001	-.8714	-.8877 (.0056)	.0161 (2.86)	.0159	.0016
$\ln \hat{\lambda}_u$	.6094	.5597 (.0390)	.0497 (1.28)	.7641	.73	.2189	.1517 .0273	.0672 (2.49)	.3721	1.0
$\hat{\lambda}_u$	1.8394	2.5647 (.1230)	.7253 (8.39)	7.5631	.0001	1.2447	1.3966 (.0399)	.1519 (3.806)	.7971	.0001
$\tilde{\lambda}_u$	1.8394	1.8173 (.0865	.0221 (.26)	3.7407	.0001	1.2447	1.1446 (.0335)	.1000 (2.859)	.5594	.0001
$\hat{\mu}_u$	.5437	.8469 (.0445)	.3032 (6.82)	.9896	.0001	.8034	1.0351 (.0306)	.2317 (7.57)	.4985	.0001
$\tilde{\mu}_u$	.5437	.5594 (.0239)	.0157 (.66)	.2867	.0001	.8034	.8320 .0236)	.0286 (1.21)	.2790	.0001
$\hat{\sigma}^2_u$	.6717	.7287 (.0096)	.0570 (5.95)	.0456	.0001	.3891	.1458 (.0038)	.0266 (8.87)	.0046	.0001

Table 5

Censoring Scheme IV

	Component 1					Component 2				
	True Value of Parameter	Mean	Bias	Var	Normal	True Value of Parameter	Mean	Bias	Var	Normal
$\hat{A}$	-1.0	-1.3776	.3776	10.4405	.59	-3.0	-2.976	.024	4.9545	1.0
		(.1445)	(2.61)				.0995	(.24)		
$\hat{B}$	1.0	1.0822	.0822	1.4731	1.0	2.0	1.9608	.0392	.678	.84
		(.0543)	(1.51)				(.0368)	(1.06)		
$\hat{V}(\hat{A})$	8.5087	11.2201	2.7114	109.119	.0001	4.750	4.9610	.211	1.348	.0001
		(.4670)	(5.81)				(.0519)	(4.07)		
$\hat{V}(\hat{B})$	1.2287	1.6055	.3768	1.8410	.0001	.6464	.6752	.0288	.0211	.0001
		(.0607)	(6.21)				(.0065)	(4.43)		
$\hat{V}(\hat{B}, \hat{A})$	-3.2150	-4.2206	1.0059	14.0076	.0001	-1.743	-1.8206	.0776	.1675	.0001
		(.1673)	(6.01)				(.0183)	(4.24)		
$\ln \hat{\lambda}_u$	.6094	.3641	.2453	1.718	.2607	.2189	.1797	.0392	.8416	.909
		(.2653)	(4.34)				(.0410)	(.96)		
$\hat{\lambda}_u$	1.8394	3.0967	1.2570	35.1877	.0001	1.2447	1.8137	.5690	5.437	.0001
		(.2653)	(4.34)				(.1040)	(5.47)		
$\tilde{\lambda}_u$	1.8394	1.4905	.3488	7.2648	.0001	1.2447	1.2413	.0037	2.873	.0001
		(.1205)	(2.89)				(.0758)	(.05)		
$\hat{\mu}_u$	.5437	4.7719	4.2282	4159.18	.0001	.8034	1.2994	.4960	2.464	.0001
		(2.8840)	(1.46)				(.0702)	(.71)		
$\tilde{\mu}_u$	.5437	.5566	.0129	.7334	.0001	.8034	.8017	.0013	.7480	.0001
		(.0383)	(.34)				(.0387)	(.03)		
$\hat{\sigma}^2_u$	1.3435	1.7932	.0497	3.7343	.0001	.7780	.8500	.0720	.0407	.0001
		(.0864)	(.58)				(.0099)	(7.27)		

Table 6

Censoring Scheme V

	Component 1					Component 2				
	True Value of Parameter	Mean	Bias	Var	Normal	True Value of Parameter	Mean	Bias	Var	Normal
$\hat{A}$	-1.0	-1.1598 (.0956)	.1598 (1.67)	4.5657	.862	-3	-3.0257 (.0737)	.0257 (.35)	2.7150	.455
$\hat{B}$	1.0	1.0425 (.0360)	.0425 (1.18)	.6496	.947	2	2.0064 (.0273)	.0064 (.23)	.3713	.573
$\hat{V}(\hat{A})$	4.1700	4.7316 (.0662)	.5616 (8.47)	2.1913	.0001	2.3856	2.4813 (.0233)	.0957 (4.11)	.0233	.0001
$\hat{V}(\hat{B})$	.6031	.6830 (.0095)	.0799 (8.41)	.0455	.0001	.3249	.3377 (.3030)	.0128 (4.26)	.0045	.0001
$\hat{V}(\hat{B}, \hat{A})$	-1.5765	-1.7873 (.0249)	.2108 (8.46)	.3109	.0001	-.8757	-.9104 (.0083)	.0347 (4.18)	.0348	.0001
$\ell n \hat{\lambda}_u$	.6094	.5181 (.0385)	.0913 (2.37)	.7411	.779	.2189	.2036 (.0304)	.0153 (.50)	.4622	.5846
$\hat{\lambda}_u$	1.8394	2.3313 (.0867)	.4919 (5.67)	3.2589	.0001	1.2447	1.5240 (.0475)	.2793 (5.88)	1.1280	.0001
$\tilde{\lambda}_u$	1.8394	1.6728 (.0651)	.1666 (2.56)	2.1204	.0001	1.2447	1.2561 (.0407)	.0114 (.28)	.8315	.0001
$\hat{\mu}_u$	.5437	.9100 (.0550)	.3663 (6.66)	1.514	.0001	.8034	.9100 (.0550)	.1066 (1.93)	1.5137	.0001
$\tilde{\mu}_u$	.5437	.5807 (.0285)	.0370 (1.30)	.4070	.0001	.8034	.8231 (.0296)	.0197 (.67)	.4377	.0001
$\hat{\sigma}_u$	.6740	.7478 (.0109)	.0738 (6.77)	.0592	.0001	.4084	.4254 (.0043)	.017 (3.95)	.0094	.0001

Table 7

Censoring Scheme VI

	Component 1					Component 2				
	True Value of Parameter	Mean	Bias	Var	Normal	True Value of Parameter	Mean	Bias	Var	Normal
$\hat{A}$	-1.0	-1.3115	.3115	12.522	.118	-3	-3.1109	.1109	6.1581	.1760
		(.1583)	(1.97)				(.1110)	(1.00)		
$\hat{B}$	1.0	1.0715	.0715	1.7535	.218	2	2.0223	.0223	.8292	.6622
		(.0592)	(1.21)				(.0407)	(.55)		
$\hat{V}(\hat{A})$	8.9299	13.3662	4.4363	181.617	.0001	5.1163	5.9371	.8208	6.8142	.0001
		(.6027)	(7.36)				(.1167)	(7.03)		
$\hat{V}(\hat{B})$	1.2809	1.8959	.6150	2.9162	.0001	.6939	.8018	.1079	.1079	.0001
		(.0764)	(8.05)				(.0147)	(7.34)		
$\hat{V}(\hat{B},\hat{A})$	-3.3588	-5.0066	-1.6478	22.7437	.0001	-1.8744	-2.1709	.2965	.8511	.0001
		(.2133)	(7.73)				(.0413)	(7.18)		
$\ln \hat{\lambda}_u$	.6094	.4130	.1964	2.0743	.025	.2189	.1439	.0750	1.0688	.010
		.0644	(3.05)				(.0422)	(1.78)		
$\hat{\lambda}_u$	1.8393	3.4528	1.6135	41.7051	.0001	1.2477	1.7861	.5414	2.8247	.0001
		(.2888)	(5.59)				(.0752)	(7.20)		
$\tilde{\lambda}_u$	1.8393	1.5791	.2602	6.9532	.0001	1.2477	1.4876	.0571	1.4911	.0001
		(.1179)	(2.21)				(.0546)	(1.05)		
$\hat{\mu}_u$	.5437	5.4206	4.8769	1580.64	.0001	.8034	1.7481	.9447	19.2456	.0001
		(1.778)	(2.74)				(.1962)	(4.81)		
$\tilde{\mu}_u$	.5437	.4494	.0943	.6336	.0001	.8034	.8089	.0055	1.1809	.0001
		(.0356)	(2.65)				(.0486)	(.11)		
$\hat{\sigma}^2$	1,4362	2.1615)	.7253	6.6171	.0001	.8804	1.0262	.1458	.2453	.0001
		(.1150)	(6.31)				(.0221)	(6.63)		